**AIMMS Modeling Guide - Portfolio Selection Problem**

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# Part V

# Advanced Optimization Modeling Applications

# Chapter 18

# A Portfolio Selection Problem

*This chapter*

In this chapter you will find an extensive introduction to portfolio selection decision making. Decision making takes place at two distinct levels. At the strategic level, the total budget to be invested is divided among major investment categories. At the tactical level, the budget for a particular investment category is divided among individual securities. Both the strategic and the tactical portfolio selection problems are considered and subsequently translated into quadratic programming models using the variance of the portfolio as a measure of risk. The objective function of the relatively small strategic portfolio selection model minimizes added covariances, which are estimated outside the model. The objective function of the tactical portfolio selection model also minimizes added covariances, but their values are not explicit in the model. Instead, scenario data is used to represent covariances indirectly, thereby avoiding the explicit construction of a large matrix. The required mathematical derivations for both models are presented in separate sections. In the last part of the chapter you will find a section on one-sided variance as an improved measure of risk, a section on the introduction of logical constraints, and a section on the piecewise linear approximation of the quadratic objective function to keep the model with logical constraints within the framework of mixed-integer linear programming.

*References*

The methodology for portfolio selection problems dates back to the work of Markowitz [Ma52] and is also discussed in [Re89].

*Keywords*

Integer Program, Quadratic Program, Mathematical Derivation, Mathematical Reformulation, Logical Constraint, Piece-Wise Approximation, Worked Example.

## 18.1 Introduction and background

*Investors*

The term investor is used for a person (or institution) who treasures a certain capital. There are many types of investors, such as private investors, institutional investors (pension funds, banks, insurance companies), governments, professional traders, speculators, etc. Each investor has a personal view of risk and reward. For instance, the point of view of a pension fund's portfolio manager will differ from that of a private investor. This is due to several reasons. The pension fund manager invests for a large group of employees of a company (or group of companies) who expect to receive their pension payments in due time. In view of this obligation, the fund manager needs to be extremely risk averse. In contrast, a private investor is only responsible for his own actions and has full control over the investment policy. He will decide on the amount of risk that he is willing to consider on the basis of personal circumstances such as his age, family situation, future plans, his own peace of mind, etc.

*Investment categories*

There are many types of investment categories. Typical examples are deposits, saving accounts, bonds, stocks, real estate, commodities (gold, silver, oil, potatoes, pigs), foreign currencies, derivatives (options, futures, caps, floors), etc. Each investment category has its own risk-reward characteristics. For instance, saving accounts and bonds are examples of low risk investment categories, whereas stocks and real estate are relatively more risky. A typical example of an investment category with a high level of risk is the category of financial derivatives such as options and futures. These derivatives are frequently used for speculation, since they offer the possibility of a high rate of return in comparison to most other investment categories.

*Discrepancies within investment categories*

The above statements need to be understood as general remarks concerning the average performance of the investment categories. Naturally, within each category, there are again discrepancies in risk-reward patterns. In the case of bonds, there are the so-called triple A bonds with a very small chance of default (for instance government loans of the United States of America). On the other end of the spectrum there are the high-yield or so-called junk bonds. These bonds are considered to have a relatively large chance of default. For instance, during the economic crises in South East Asia many of the government bonds issued by countries from that area are considered to be junk bonds. The interest paid on these type of bonds can amount to 30 percent. Similarly in the category of stocks there are many different types of stocks. The so-called blue chips are the stocks corresponding to the large international companies with a reliable track record ranging over a long period of time. On the other hand, stocks of relatively new and small companies with high potential but little or no profits thus far, often have a high expected return and an associated high degree of risk.

*Securities are investment products*

The term security is used to denote one particular investment product within an investment category. For instance, the shares of companies like IBM or ABN AMRO are examples of securities within the category of stocks. Similarly a 2002 Oct Future contract on the Dutch Index of the Amsterdam Exchange (AEX) is an example of a security within the category of derivatives.

*Percentage rate of return is widely used*

Investing in a portfolio requires funds and thus a budget. Funds are usually measured in some monetary unit such as dollars, yens or guilders. Using absolute quantities, such as \$-returns and \$-investments, has the drawback of currency influences and orders of magnitude. That is why the percentage rate of return (in the sequel referred to as 'rate of return'),

$$100 \cdot \frac{\text{new return} - \text{previous return}}{\text{previous return}}$$

is a widely accepted measure for the performance of an investment. It is dimensionless, and simplifies the weighting of one security against the other. As will be discussed later, the choice of the time step size between subsequent returns has its own effect on the particular rate of return values.

*Concept of risk*

Naturally, investors would like to see high rates of return on their investments. However, holding securities is risky. The value of a security may appreciate or depreciate in the market, yielding a positive or negative return. In general, one can describe risk as the uncertainty regarding the actual rate of return on investment. Since most investors are risk averse, they are only willing to accept an additional degree of risk if the corresponding expected rate of return is relatively high.

*Portfolio diversification*

Instead of investing in one particular security, most investors will spread their funds over various investments. A collection of securities is known as a *portfolio*. The rationale for investing in a portfolio instead of a single security is that different securities perform differently over time. Losses on one security could be offset by gains on another. Hence, the construction of a portfolio enables an investor to reduce his overall risk, while maintaining a desired level of expected return. The concept of investing in a number of different securities is called *diversification*. This concept and its mathematical background was introduced by H. Markowitz in the early fifties ([Ma52]).

*Practical limitations*

Although diversification is a logical strategy to reduce the overall risk of a portfolio, there will be practical obstacles in realizing a well-diversified portfolio. For instance, the budget limitation of a small private investor will severely restrict the possibilities of portfolio diversification. This is not the case for the average pension fund manager, who manages a large amount of funds. He, on the other hand, may face other restrictions due to liquidity requirements over time by existing pension holders.

*Quantification through statistical concepts*

The main focus in this chapter is on how to quantify the risk associated with a complete portfolio. What is needed, is a quantitative measure to reflect an investor's notion of uncertainty with respect to performance of the portfolio. The approach presented in this chapter is based on tools from statistics, and is one that is frequently referred to in the literature. Nevertheless, it is just one of several possible approaches.

## 18.2 A strategic investment model

*This section*

In this section a strategic portfolio selection model will be formulated. It models how top management could spread an overall budget over several investment categories. Once their budget allocation becomes available, tactical investment decisions at the next decision level must be made concerning individual securities within each investment category. Such a two-phase approach supports hierarchical decision making which is typical in large financial institutions.

*The strategic investment decision*

During the last decade there has been an enormous growth in investment possibilities. There are several logical explanations for this phenomenon. The globalization of financial markets has opened possibilities of geographical diversification. Investments in American, European or Asian stocks and bonds have completely different risk-reward patterns. The further professionalization of financial institutions has led to the introduction of all kinds of new financial products. In view of these developments, top management needs to concentrate on the global partitioning of funds into investment categories. This is referred to as the *strategic investment decision*.

*Decision variables*

The allocation of the total budget over the various investment categories will be expressed in terms of budget fractions. These fractions need to be determined, and form the set of decision variables. Each budget fraction is associated with a particular investment category, and is defined as the amount invested in this category divided by the total budget.

*Objective function*

The objective is to minimize the portfolio risk. In his paper Markowitz [Ma52] introduced the idea of using the variance of the total portfolio return as a measure of portfolio risk. His proposed risk measure has become a standard, and will also be used in this chapter.

*Minimal level of expected return*

Each category has a known level of expected return. These levels, together with the budget fractions, determine the level of expected return for the entire portfolio. The investor will demand a minimal level of expected return for the entire portfolio. This requirement forms the main constraint in the portfolio model.

*Verbal model statement*

The overall approach is to choose budget fractions such that the expected return of the portfolio is greater than or equal to some desired target, and such that the level of risk is as small as possible. The model can be summarized as follows.

**Minimize:** *the total risk of the portfolio*

**Subject to:**

- *minimal level of desired return: the expected return of the portfolio must be larger than a given minimal desired level.*
- *definition of budget fractions: all budget fractions are nonnegative, and must add to 1.*

*Notation*

The following symbols will be used.

**Index:**

| | |
|---|---|
| $j$ | *investment categories* |

**Parameters:**

| | |
|---|---|
| $R_j$ | *return of category j (random variable)* |
| $m_j$ | *expected value of random variable $R_j$* |
| $M$ | *desired (expected) portfolio return* |

**Variable:**

| | |
|---|---|
| $x_j$ | *fraction of the budget invested in category j* |

*Mathematical model statement*

The mathematical description of the model can be stated as follows.

**Minimize:**

$$\mathrm{Var}[\sum_j R_j x_j]$$

**Subject to:**

$$\sum_j m_j x_j \ge M$$

$$\sum_j x_j = 1$$

$$x_j \ge 0 \qquad \forall j$$

*Deterministic equivalent of objective*

Note that the objective function makes reference to random variables, and is therefore not a deterministic expression. To rewrite this expression, you need some mathematical concepts that are developed in the next section. It will be shown that the objective function is equivalent to minimizing added covariances. That is

$$\mathrm{Var}[\sum_j R_j x_j] = \sum_{jk} x_j \,\mathrm{Cov}[R_j, R_k] x_k$$

Here, the new deterministic equivalent of the objective is a quadratic function in terms of the unknown $x$-variables. The coefficients $\mathrm{Cov}[R_j, R_k]$ are known input data.

## 18.3 Required mathematical concepts

*This section*

In this section the statistical concepts of expectation and variance are discussed, together with their role in the portfolio selection model presented in the previous section. The corresponding statistical functions are available in AIMMS.

*The random variable R*

Consider for a moment the investment in one particular investment category, and define the random variable $R$ that describes the rate of return on this investment category after one year. For simplicity, assume that $R$ has a finite set $I$ of values (outcomes), which are denoted by $r_i$ with corresponding probabilities $p_i$ for all $i \in I$ and such that $\sum_i p_i = 1$.

*Concept of expectation*

The concept of expectation (or expected value) corresponds to your intuitive notion of average. The expected value of the random variable $R$ is denoted by $\mathrm{E}[R]$ and is defined as follows.

$$\mathrm{E}[R] = \sum_i r_i p_i$$

Note that whenever $r_i = c$ (i.e. constant) for all $i \in I$, then the expected value

$$\mathrm{E}[R] = \sum_i r_i p_i = c \sum_i p_i = c$$

is also constant. The following result will be used to advantage in various derivations throughout the remainder of this chapter. Whenever $f$ is a function of the random variable $R$, the expected value of the random variable $f(R)$ is equal to

$$\mathrm{E}[f(R)] = \sum_i f(r_i) p_i$$

*Concept of variance ...*

The concept of variance corresponds to the intuitive notion of variation around the expected value. The variance of a random variable $R$ is denoted by $\mathrm{Var}[R]$ and is defined as follows.

$$\mathrm{Var}[R] = \mathrm{E}[(R - \mathrm{E}[R])^2]$$

Using the result of the previous paragraph, this expression can also be written as

$$\mathrm{Var}[R] = \sum_i (r_i - \mathrm{E}[R])^2 p_i$$

*... as measure of risk*

Variance turns out to be a suitable measure of risk. The main reason is that variance, just like most other risk measures, is always nonnegative and zero variance is a reflection of no risk. Note that if $R$ has only one possible value $c$, then the return is always constant and thus without risk. In that case, the expected value $\mathrm{E}[R]$ is $c$ and

$$\mathrm{Var}[R] = \sum_i (R_i - c)^2 p_i = 0$$

*Chebyshev inequality ...*

A well-known inequality which is closely related to the concept of variance is the *Chebyshev* inequality:

$$P(|R - \mathrm{E}[R]| > \alpha\sigma) < \frac{1}{\alpha^2} \quad \forall\, \alpha > 0$$

In this inequality the term

$$\sigma = \sqrt{\mathrm{Var}[R]}$$

is used, and is called the *standard deviation* of the random variable $R$.

*... supports variance as risk measure*

The Chebyshev inequality states that the probability of an actual rate of return differing more than $\alpha$ times the standard deviation from its expected value, is less than 1 over $\alpha$ squared. For instance, the choice $\alpha = 5$ gives rise to a probability of at least 96 percent that the actual rate of return will be between $\mathrm{E}[R] - 5\sigma$ and $\mathrm{E}[R] + 5\sigma$. The smaller the variance, the smaller the standard deviation, and hence the smaller the distance between the upper and lower value of this confidence interval. This property of the Chebyshev inequality also supports the notion of variance as a measure of risk.

*Derivation of the Chebyshev inequality*

By straightforward use of the definition of variance the following derivation leads to the Chebyshev inequality

$$\begin{aligned}
\sigma^2 &= \mathrm{Var}[R] = \mathrm{E}[(R - \mathrm{E}[R])^2] = \sum_i (r_i - \mathrm{E}[R])^2 p_i \\
&= \sum_{i \,\big|\, |r_i - \mathrm{E}[R]| \le \alpha\sigma} (r_i - \mathrm{E}[R])^2 p_i + \sum_{i \,\big|\, |r_i - \mathrm{E}[R]| > \alpha\sigma} (r_i - \mathrm{E}[R])^2 p_i \\
&\ge \sum_{i \,\big|\, |r_i - \mathrm{E}[R]| > \alpha\sigma} (r_i - \mathrm{E}[R])^2 p_i \\
&> \sum_{i \,\big|\, |r_i - \mathrm{E}[R]| > \alpha\sigma} (\alpha\sigma)^2 p_i \\
&= \alpha^2\sigma^2 \sum_{i \,\big|\, |r_i - \mathrm{E}[R]| > \alpha\sigma} p_i \\
&= \alpha^2\sigma^2 P(|R - \mathrm{E}[R]| > \alpha\sigma)
\end{aligned}$$

Thus, in summary, $\sigma^2 > \alpha^2\sigma^2 P(|R - \mathrm{E}[R]| > \alpha\sigma)$, which immediately leads to the Chebyshev inequality.

*Expected value of portfolio return*

Suppose there are a finite number of investment categories $j$ from which an investor can select. Let $R_j$ be the random variable that denotes the rate of return on the $j$-th category with corresponding values (outcomes) $r_{ij}$ and probabilities $p_{ij}$ such that $\sum_i p_{ij} = 1 \, \forall j$. Let $x_j$ be the fraction of the budget invested in category $j$. The return of the total portfolio is then equal to $\sum_j R_j x_j$, which, being a function of random variables, is also a random variable. This implies, using the previously described identity $\mathrm{E}[f(R)] = \sum_i f(r_i) p_i$ , that the expected value of the portfolio return equals

$$\begin{aligned}
\mathrm{E}[\sum_j R_j x_j] &= \sum_i (\sum_j r_{ij} x_j p_{ij}) \\
&= \sum_j x_j (\sum_i r_{ij} p_{ij}) \\
&= \sum_j x_j \, \mathrm{E}[R_j]
\end{aligned}$$

This last expression is just the weighted sum of the expected values associated with the individual investment categories.

*Variance of portfolio return*

The variance of the portfolio return can now be determined as follows.

$$\begin{aligned}
\mathrm{Var}[\sum_j R_j x_j] &= \mathrm{E}[(\sum_j R_j x_j - \mathrm{E}[(\sum_j R_j x_j)])^2] \\
&= \mathrm{E}[(\sum_j R_j x_j - \sum_j x_j E[R_j])])^2] \\
&= \mathrm{E}[(\sum_j x_j (R_j - \mathrm{E}[R_j]))^2] \\
&= \mathrm{E}[(\sum_{jk} x_j (R_j - \mathrm{E}[R_j]) x_k (R_k - \mathrm{E}[R_k]))] \\
&= \sum_{jk} x_j x_k \, \mathrm{E}[(R_j - \mathrm{E}[R_j])(R_k - \mathrm{E}[R_k])]
\end{aligned}$$

Here, the term

$$\mathrm{E}[(R_j - \mathrm{E}[R_j])(R_k - E[R_k])]$$

is called the *covariance* of the random variables $R_j$ and $R_k$, and will be denoted by $\mathrm{Cov}[R_j, R_k]$. Note that $\mathrm{Cov}[R_j, R_j] = \mathrm{Var}[R_j]$ by definition.

*Concept of covariance*

The covariance of two random variables is a measure of the relation between above and below average values of these two random variables. When both positive and negative deviations tend to occur simultaneously, their covariance will be positive. When positive deviations of one of them tends to occur often with negative deviations of the second, their covariance will be negative. Only when positive and negative deviations occur randomly, their covariance will tend to be zero.

*The objective minimizes added covariances*

By using the covariance terms, portfolio risk can be written as

$$\mathrm{Var}[\sum_j R_j x_j] = \sum_{jk} x_j \,\mathrm{Cov}[R_j, R_k] x_k$$

Thus, the objective of the model can be formulated as the minimization of weighted covariances summed over all possible random pairs $(R_j, R_k)$.

*Diversification as a logical strategy*

From a mathematical point of view, the model shows that it is advisable to invest in categories with negative covariances. The logical explanation for this is that below average results of one investment category are likely to be offset by above average results of the other. Hence, the model formulation using covariances can be seen as the formalization of the intuitive concept of spreading risk by using various securities.

## 18.4 Properties of the strategic investment model

*This section*

In this section several mathematical properties of the strategic investment model are investigated. In summary, it is shown that $(a)$ any optimal solution of the nonlinear programming model developed in Section 18.2 is also a global optimum, $(b)$ the risk-reward curve is nondecreasing and convex, and $(c)$ multiple optimal portfolio returns are perfectly correlated.

*Optimum is global*

Any optimal solution of the presented portfolio selection model is globally optimal. This follows from the theory of optimization. The theory states that, whenever a model has linear constraints and the objective function $f(x)$ to be minimized is convex, i.e.

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \quad \forall \alpha \in [0,1]$$

then any optimal solution of the model is also a globally optimal solution. In the portfolio selection model the objective function

$$f(x) = \sum_{jk} x_j \,\mathrm{Cov}[R_j, R_k] x_k$$

is a quadratic function, and is convex if and only if the associated matrix of second-order derivatives is positive semi-definite.

*Required matrix condition is satisfied*

Note that the matrix with second-order derivatives is precisely the covariance matrix with elements $\mathrm{Cov}[R_j, R_k]$. Such a matrix is positive semi-definite if and only if

$$\sum_{jk} x_j \,\mathrm{Cov}[R_j, R_k] x_k \geq 0 \quad \forall x_j, x_k \in \mathbb{R}$$

This mathematical condition, however, happens to be equivalent to the inequality $\mathrm{Var}[\sum_j R_j x_j] \geq 0$, which is true by definition.

*Parametric risk-reward curve $V(M)$*

The input parameter $M$ characterizes the individual preference of the investor. Instead of solving the model for one particular value of $M$, it will be interesting for an investor to see what the changes in optimal risk value will be as the result of changes in $M$. The optimal risk value is unique for each value of $M$ due to the global optimality of the solution. Thus the optimal risk value $V$ can be considered as a function of the desired minimal level of expected return $M$. This defines a parametric curve $V(M)$.

*Practical levels of desired return*

The value $V(M)$ is defined for all values of $M$ for which the model is solvable. It is straightforward to verify that $-\infty \le M \le \max_j m_j \equiv M_{\max}$. When $M$ is set to $-\infty$, $V(M)$ obtains its lowest value. An investor will be interested in the largest feasible value of $M$ that can be imposed such that $V(M)$ remains at its lowest level. Let $M_{\min}$ be this level of $M$. Then, for all practical purposes, $M$ can be restricted such that $M_{\min} \le M \le M_{\max}$. Note that the value of $M_{\min}$ can be determined experimentally by solving the following penalized version of the portfolio selection model

**Minimize:**

$$\mathrm{Var}[\sum_j R_j x_j] - \lambda(\sum_j m_j x_j)$$

**Subject to:**

$$\sum_j x_j = 1$$

$$x_j \ge 0 \qquad \forall j$$

for a sufficiently small value of $\lambda > 0$.

*Properties of risk-reward curve $V(M)$*

The optimal value $V(M)$ is nondecreasing in $M$, because any feasible solution of the model for a particular value of $M$ will also be a solution for smaller values of $M$. In addition, it will be shown in the paragraph below that $V(M)$ is convex. These two properties (nondecreasing and convex), coupled with the definition of $M_{\min}$ from the previous paragraph, imply that $V(M)$ is strictly increasing on the interval $[M_{\min}, M_{\max}]$.

*Proof that $V(M)$ is convex*

Let $M = \alpha M_1 + (1-\alpha)M_2$ for $\alpha \in [0,1]$, and let $M_{\min} \le M_1 \le M_2 \le M_{\max}$. In addition, let $x_M$, $x_1$ and $x_2$ be the optimal solutions corresponding to $M$, $M_1$ and $M_2$, respectively. Furthermore, let $\mathbf{Q}$ denote the covariance matrix. Then, as explained further below,

$$\begin{aligned}
V(M) &= x_M^T \, \mathbf{Q} \, x_M \\
&\le (\alpha x_1 + (1-\alpha)x_2)^T \, \mathbf{Q} \, (\alpha x_1 + (1-\alpha)x_2) \\
&\le \alpha x_1^T \, \mathbf{Q} \, x_1 + (1-\alpha) x_2^T \, \mathbf{Q} \, x_2 \\
&= \alpha V(M_1) + (1-\alpha) V(M_2)
\end{aligned}$$

The inequality on the second line of the proof follows from the fact that $\alpha x_1 + (1-\alpha)x_2$ is a feasible but not necessarily optimal solution for $M = \alpha M_1 + (1-\alpha)M_2$. The third line follows directly from the convexity of the quadratic objective function, which then establishes the desired result that

$$V(\alpha M_1 + (1-\alpha)M_2) \leq \alpha V(M_1) + (1-\alpha)V(M_2)$$

Thus, $V(M)$ is convex in $M$.

*Multiple optimal portfolios ...*

Consider a fixed value of $M$ and two distinct optimal portfolios $x_1^*$ and $x_2^*$ with equal variance $V^*(M)$. Then any convex combination of these two portfolios will also be a feasible portfolio due to the linearity of the model constraints. From the convexity of the quadratic objective function the variance of each intermediate portfolio return can only be less than or equal to $V^*(M)$. However, it cannot be strictly less than $V^*(M)$, because this would contradict the optimality of $V^*(M)$. Hence the variance of the return of each intermediate portfolio is equal to $V^*(M)$ and thus also optimal.

*... have perfectly correlated returns*

As will be shown next, any two distinct optimal portfolios $x_1^*$ and $x_2^*$ for a fixed value of $M$ have perfectly correlated returns. Let $P_1$ and $P_2$ be the corresponding portfolio returns, and consider the variance of the return of an intermediate portfolio. This variance can be written as a weighted sum of individual covariances as follows.

$$\begin{aligned}
&\mathrm{Var}[\alpha P_1 + (1-\alpha)P_2] \\
&\quad = \alpha^2 \,\mathrm{Cov}[P_1, P_1] + 2\alpha(1-\alpha)\,\mathrm{Cov}[P_1, P_2] + (1-\alpha)^2\,\mathrm{Cov}[P_2, P_2] \\
&\quad = \alpha^2 \,\mathrm{Var}[P_1] + (1-\alpha)^2\,\mathrm{Var}[P_2] + 2\alpha(1-\alpha)\,\mathrm{Cov}[P_1, P_2]
\end{aligned}$$

From the previous paragraph it is also known that

$$\mathrm{Var}[\alpha P_1 + (1-\alpha)P_2] = \mathrm{Var}[P_1] = \mathrm{Var}[P_2]$$

Therefore, by substituting the above identities, the term $\mathrm{Cov}[P_1, P_2]$ can be determined as follows.

$$\mathrm{Cov}[P_1, P_2] = \frac{1-\alpha^2-(1-\alpha)^2}{2\alpha(1-\alpha)}\,\mathrm{Var}[P_1] = \mathrm{Var}[P_1]$$

This implies that the correlation coefficient $\rho$ between the portfolio returns $P_1$ and $P_2$, defined as $\mathrm{Cov}[P_1, P_2]/(\sqrt{\mathrm{Var}[P_1]\,\mathrm{Var}[P_2]})$, is equal to 1. Hence, $P_1$ and $P_2$ are perfectly correlated.

*Illustrating global optimality*

Figure 18.1 gives an example of the feasible region and contours of the objective function for a portfolio of two securities. Notice that the feasible region is now restricted to that part of the budget line $x_1 + x_2 = 1$ for which the target return is at least achieved. It is intuitively clear that the optimal combination of securities is globally optimal, due to the shape of the contours of the objective.

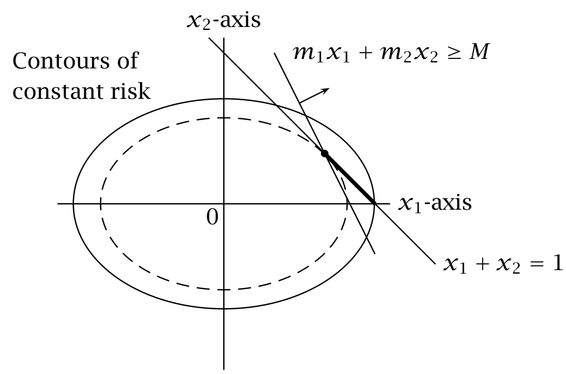


Figure 18.1: The feasible region and two objective function contours

## 18.5 Example of strategic investment model

*This section*

In this section a small example of the strategic investment approach is presented. The required input data at this strategic level is usually not directly obtainable from public sources such as stock exchanges etc. Instead such input data is estimated from statistical data sources in a nontrivial manner, and is provided in this example without any further explanation.

*Data*

Consider three investment categories: stocks, bonds and real estate. The corresponding random variables will be denoted by $X_1, X_2$ and $X_3$. The minimal level of expected return $M$ will be set equal to 9.0. The expected return values, together with the covariances between investment categories, are displayed in Table 18.1.

| $i$ | $m_i$ | $\mathrm{Cov}[X_i, X_j]$ | | |
|---|---|---|---|---|
| | $j$ | 1 | 2 | 3 |
| 1 | 10.800 | 2.250 | −0.120 | 0.450 |
| 2 | 7.600 | −0.120 | 0.640 | 0.336 |
| 3 | 9.500 | 0.450 | 0.336 | 1.440 |

Table 18.1: Expected returns and covariances

*The optimal solution . . .*

After solving the portfolio model described in Section 18.2, the optimal portfolio fractions are $x_1 = 0.3233, x_2 = 0.4844, x_3 = 0.1923$. Therefore, approximately 32 percent will be invested in stocks, 48 percent in bonds and 19 percent in real estate. The corresponding optimal portfolio risk is equal to 0.5196.

*... supports diversification*

Note that the optimal portfolio risk (approximately 0.52) is even smaller than the variance of bonds (0.64), which is the lowest variance associated with any particular investment category. In addition, note that the expected portfolio return (9.00) is higher than if the entire investment had been in bonds only (namely 7.6). These results clearly illustrate the benefits of portfolio diversification.

*Risk-reward characteristic*

It is of interest to an investor to see what the change in optimal value $V$ will be as a consequence of changing the value of desired return $M$. Below the function $V(M)$ is presented on the interval $[7.6, 10.8]$.

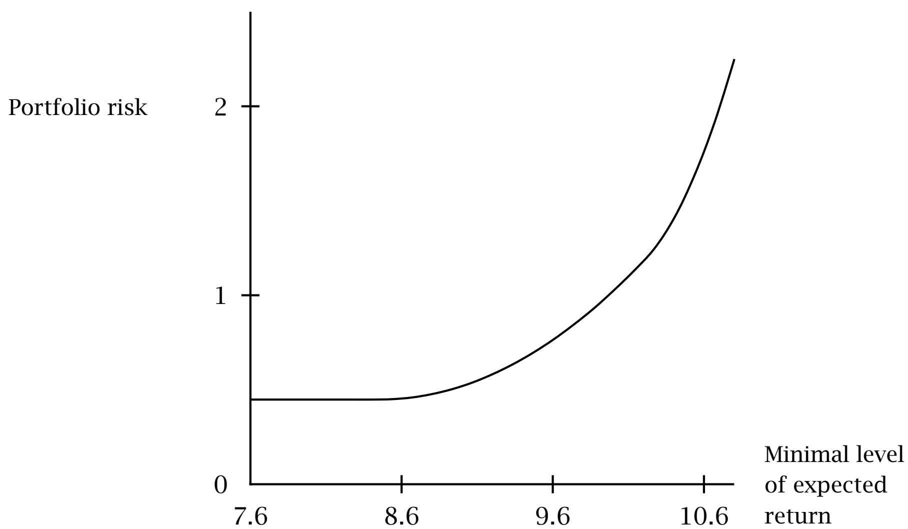


Figure 18.2: Risk-reward characteristic

The smallest level of expected return that is of interest to the investor is $M_{\min} = 8.4$, which can be derived by solving the model with the penalized objective function from the previous section. Note that this value is larger than $\min_i m_i = 7.6$. For values of $M$ greater than $M_{\min}$, the curve is strictly increasing and the constraint regarding the minimal level of expected return is binding. Based on this curve an investor can make his trade-off between risk and reward.

*Unique optimal portfolios*

As explained in the previous section, optimal portfolios need not be unique. In this example, however, they are unique, because there are no perfectly correlated portfolios. You may verify this observation by computing the correlation coefficients between returns on the basis of the covariances listed in Table 18.1.

*Portfolio diversification illustrated*

Similar to the parametric curve representing the optimal risk value as a function of the minimal level of expected return, you may also want to view the budget fractions of the optimal portfolio as functions of the minimal level of expected return. These parametric curves illustrate portfolio diversification, and are presented in Figure 18.3.

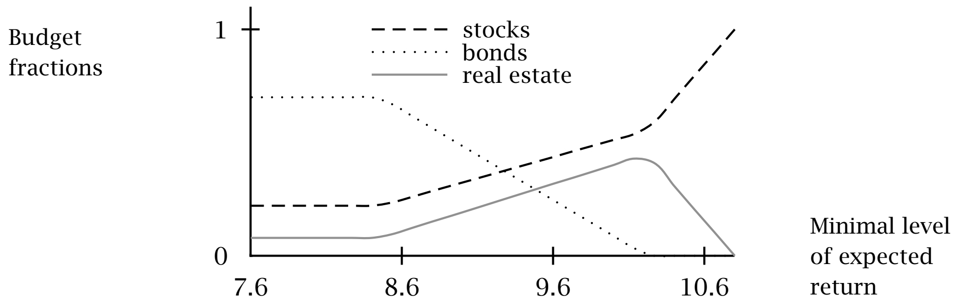


Figure 18.3: Portfolio diversification

## 18.6 A tactical investment model

*This section*

At the tactical level, there are specialized fund managers to manage a particular investment category. Each manager receives a specific budget, which is based on the solution of the strategic investment model. In this section the tactical investment model is derived from the strategic investment model. The major difference between the two models is that the much larger covariance matrix in the tactical model is no longer modeled as an explicit matrix.

*From aggregated investment categories ...*

The solution of the strategic investment model in the previous section suggested to invest approximately 32 percent of the overall budget in stocks. Such a result was based on aggregated data representing the entire stock investment category, and not on data regarding individual stocks. For individual stocks the question naturally arises which ones should be selected to spend the 32 percent of the total budget. This is considered to be a question at the tactical level.

*... to numerous individual securities*

The possibilities to select individual securities from within a particular investment category are enormous. If the underlying decision model at this level was the same as the strategic decision model, the corresponding covariance matrix would become very large. That is why an alternative modeling approach is proposed to deal with securities at the tactical level

*Observed returns determine scenarios . . .*

In the paragraphs to follow, it is shown that the original portfolio variance as measure of risk can be estimated directly from real-world observed returns taken at distinct points in time. Each time observation consists of a vector of returns, where the size of the vector is equal to the number of individual securities. By considering two subsequent time periods, it is possible to compute the corresponding rate of returns. The resulting vector is referred to as a *scenario*.

*. . . based on specific time steps*

By construction, there are as many scenarios as there are time observations minus one. It is to be expected that the time step size will vary the rate of return values associated with each scenario. The movements in returns between subsequent time observations are likely to be different when you consider hourly, daily or monthly changes in return values. The choice of time step in the computation of scenarios should be commensurable with the time unit associated with the investment decision.

*Notation*

Consider a vector of random variables denoting rates of returns $R_j$ for $j$. Every instance of this vector denotes a scenario. Assume that there is a finite set of scenarios. Let $T$ denote this set with index $t \in T$. Let $r_t$ denote a single scenario, and $p(r_t)$ its associated probability. By definition, the sum over all scenarios of the probabilities (i.e. $\sum_t p(r_t)$) equals 1.

**Index:**

| | |
|---|---|
| $t$ | *scenarios of size* $\lvert T \rvert$ |

**Parameters:**

| | |
|---|---|
| $r_t$ | *vector of particular return rates for scenario t* |
| $p(r_t)$ | *probability of scenario t* |
| $r_{tj}$ | *particular return rate of security* $j$ *in scenario t* |

Note that the symbol $r$ is overloaded in that it is used for both the vector of return rates ($r_t$) and the individual return rates per scenario ($r_{tj}$). This is done to resemble previous notation and is used throughout the remainder of this section for consistency.

*Reformulation of the objective*

Consider the following straightforward algebraic manipulations based on moving the $\sum$-operator, and using the properties of the $E$-operator.

$$
\begin{aligned}
\mathrm{Var}[\sum_j R_j x_j] &= \mathrm{E}[((\sum_j R_j x_j) - \mathrm{E}[\sum_j R_j x_j])^2] \\
&= \mathrm{E}[((\sum_j R_j x_j) - \sum_j x_j \, \mathrm{E}[R_j])^2] \\
&= \mathrm{E}[(\sum_j x_j (R_j - \mathrm{E}[R_j]))^2] \\
&= \sum_t (\sum_j x_j (r_{tj} - \mathrm{E}[R_j]))^2 p(r_t) \\
&= \sum_t p(r_t) y_t^2
\end{aligned}
$$

where $y_t = \sum_j x_j (r_{tj} - \mathrm{E}[R_j]) \quad \forall t \in T$. This results in a compact formula for the objective function in terms of the new variables $y_t$. These new decision variables plus their definition will be added to the model.

*Comparing the number of nonlinear terms*

The repetitive calculation of the objective function and its derivatives, required by a nonlinear programming solver, can be carried out much faster in the above formulation than in the formulation of Section 18.2. This is because, in the tactical investment model, $|T|$ (the number of scenarios) is typically much smaller than $|J|$ (the number of individual securities). Therefore, the number of nonlinear terms ${y_t}^2$ is significantly smaller than the number of nonlinear terms $x_j x_k$.

*The model formulation*

Let $m_j = \mathrm{E}[R_j]$ be the expected value of security $j$, and let $d_{tj} = (r_{tj} - m_j)$ be the deviation from the expected value defined for each scenario. Then using the approach presented in the previous paragraph, a quadratic optimization model with $x_j$ and $y_t$ as the decision variables can be written as follows.

**Minimize:**

$$\sum_t p(r_t) y_t^2$$

**Subject to:**

$$
\begin{aligned}
\sum_j d_{tj} x_j &= y_t \qquad \forall t \\
\sum_j m_j x_j &\geq M \\
\sum_j x_j &= 1 \\
x_j &\geq 0 \qquad \forall j
\end{aligned}
$$

*Model properties unchanged*

The properties of the above investment model are the same as the ones associated with the strategic investment model, because the above model is just an equivalent reformulation of the model presented in Section 18.2. Of course, it is still possible to derive the model properties directly from the mathematical formulation above. For instance, the verification that the new objective function is also convex, follows directly from the observation that the matrix with second-order derivatives is a $|T| \times |T|$ diagonal matrix with $2p(r_t) \geq 0$ as its diagonal elements. Such a matrix is always positive semi-definite.

## 18.7 Example of tactical investment model

*This section*

In this section you find a small example in terms of 5 individual stocks and 51 observations. In a realistic application, the number of observations is usually in the hundreds, while the number of candidate stocks is likely to be a multiple thereof.

*Data*

The stocks that can be part of the portfolio are: RD (Royal Dutch), AKZ (Akzo Nobel), KLM (Royal Dutch Airline Company), PHI (Philips) and UN (Unilever). The historical data are weekly closing values from August 1997 to August 1998, and are provided in Table 18.2. The corresponding weekly rates of return can be computed on the basis of these return values, and have all been given equal probability.

*The results*

As for the strategic investment model a risk-reward characteristic can be presented. The expected level of return for the various stocks is: RD -0.28, AKZ 0.33, KLM 0.40, PHI 0.30, UN 0.55. The parametric curve $V(\mu)$ is computed on the interval $[0, 0.55]$. Below both the risk-reward characteristic as well as the budget fractions of the optimal solutions are presented in Figure 18.4 and Figure 18.5.

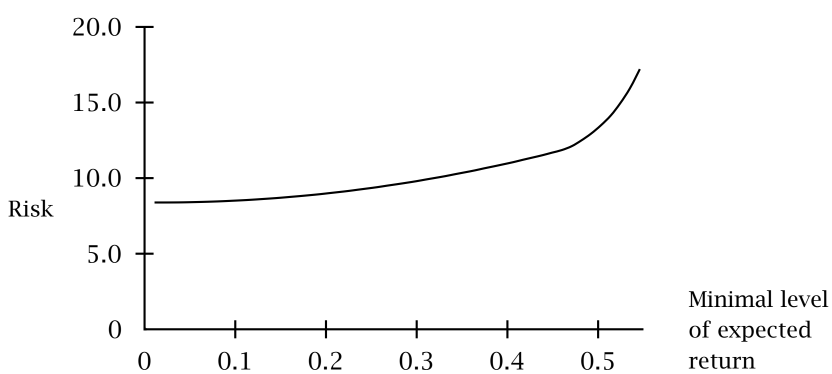


Figure 18.4: Risk-reward characteristic

| **Scena–rios** | **Valueofstocks** *RD* | *AKZ* | *KLM* | *PHI* | *UN* | **Scena–rios** | **Valueofstocks** *RD* | *AKZ* | *KLM* | *PHI* | *UN* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t-0$ | 111.0 | 82.5 | 70.0 | 154.6 | 110.8 | $t-26$ | 112.8 | 107.0 | 76.3 | 155.9 | 134.2 |
| $t-1$ | 108.1 | 81.6 | 73.7 | 152.4 | 108.0 | $t-27$ | 109.7 | 110.4 | 86.0 | 155.0 | 140.9 |
| $t-2$ | 107.9 | 80.1 | 72.3 | 146.1 | 103.7 | $t-28$ | 111.7 | 109.7 | 88.9 | 149.9 | 138.2 |
| $t-3$ | 108.5 | 83.1 | 69.7 | 157.5 | 106.6 | $t-29$ | 120.4 | 105.9 | 83.5 | 153.5 | 141.6 |
| $t-4$ | 111.4 | 85.0 | 69.5 | 168.4 | 107.3 | $t-30$ | 118.0 | 105.9 | 83.4 | 153.0 | 140.6 |
| $t-5$ | 115.5 | 92.6 | 74.8 | 166.9 | 109.5 | $t-31$ | 119.7 | 103.0 | 84.9 | 149.9 | 158.2 |
| $t-6$ | 113.2 | 91.6 | 73.8 | 164.1 | 108.7 | $t-32$ | 116.7 | 102.4 | 84.9 | 153.7 | 149.6 |
| $t-7$ | 111.9 | 88.3 | 70.2 | 169.0 | 111.1 | $t-33$ | 115.8 | 107.2 | 86.1 | 167.0 | 152.8 |
| $t-8$ | 99.7 | 80.8 | 64.3 | 143.8 | 101.0 | $t-34$ | 113.7 | 104.5 | 78.9 | 181.0 | 144.3 |
| $t-9$ | 105.1 | 86.1 | 71.8 | 151.3 | 105.4 | $t-35$ | 115.7 | 105.8 | 79.5 | 189.4 | 155.5 |
| $t-10$ | 100.9 | 81.8 | 71.7 | 148.3 | 109.6 | $t-36$ | 114.4 | 104.8 | 79.1 | 197.9 | 154.2 |
| $t-11$ | 105.0 | 85.6 | 69.5 | 140.6 | 112.8 | $t-37$ | 113.8 | 103.8 | 79.9 | 201.7 | 154.5 |
| $t-12$ | 105.2 | 84.6 | 70.5 | 131.5 | 113.6 | $t-38$ | 114.0 | 107.0 | 82.0 | 196.3 | 158.0 |
| $t-13$ | 107.0 | 90.3 | 74.9 | 138.0 | 117.4 | $t-39$ | 114.1 | 107.4 | 77.2 | 188.0 | 163.8 |
| $t-14$ | 109.0 | 88.3 | 78.5 | 135.4 | 123.1 | $t-40$ | 111.5 | 112.0 | 78.5 | 189.9 | 164.3 |
| $t-15$ | 111.4 | 85.6 | 73.0 | 114.0 | 124.5 | $t-41$ | 109.2 | 106.8 | 77.1 | 172.1 | 163.9 |
| $t-16$ | 107.2 | 81.6 | 74.5 | 116.1 | 118.7 | $t-42$ | 110.1 | 105.3 | 76.1 | 178.0 | 165.6 |
| $t-17$ | 111.3 | 87.4 | 75.0 | 121.6 | 125.0 | $t-43$ | 112.8 | 113.1 | 82.6 | 171.0 | 161.4 |
| $t-18$ | 108.6 | 87.5 | 77.0 | 127.6 | 127.7 | $t-44$ | 111.0 | 116.6 | 91.4 | 179.5 | 165.4 |
| $t-19$ | 105.6 | 86.6 | 72.4 | 116.2 | 121.2 | $t-45$ | 105.6 | 126.7 | 94.5 | 180.6 | 160.9 |
| $t-20$ | 105.9 | 90.0 | 71.4 | 128.6 | 125.1 | $t-46$ | 107.3 | 123.1 | 90.0 | 173.5 | 162.0 |
| $t-21$ | 104.7 | 91.4 | 68.9 | 129.4 | 119.9 | $t-47$ | 103.2 | 112.3 | 88.5 | 164.4 | 153.0 |
| $t-22$ | 107.7 | 95.3 | 69.7 | 137.1 | 117.9 | $t-48$ | 102.8 | 103.1 | 81.8 | 164.2 | 141.2 |
| $t-23$ | 107.4 | 92.9 | 68.6 | 134.5 | 124.6 | $t-49$ | 93.9 | 95.0 | 80.7 | 153.0 | 133.4 |
| $t-24$ | 108.0 | 97.0 | 70.2 | 156.0 | 124.3 | $t-50$ | 93.6 | 92.7 | 80.5 | 164.0 | 139.3 |
| $t-25$ | 104.7 | 102.6 | 74.3 | 159.5 | 128.8 | | | | | | |

Table 18.2: Stock returns for 50 different scenarios

*Comment*

Although the stock RD has a negative expected return, it is a major component in the optimal portfolio for low values of minimal expected return. This unexpected result is an artifact of the problem formulation and is addressed in the next section. The stock is included due to its relative stable behavior which stabilizes the overall performance of the portfolio and is therefore used to reduce the overall risk of the portfolio. Only for large values of minimal expected rates of return (over the 0.02 on a weekly basis, so over the 10 percent on a yearly basis) the budget fraction of the stock UN will be larger than the fraction of RD.

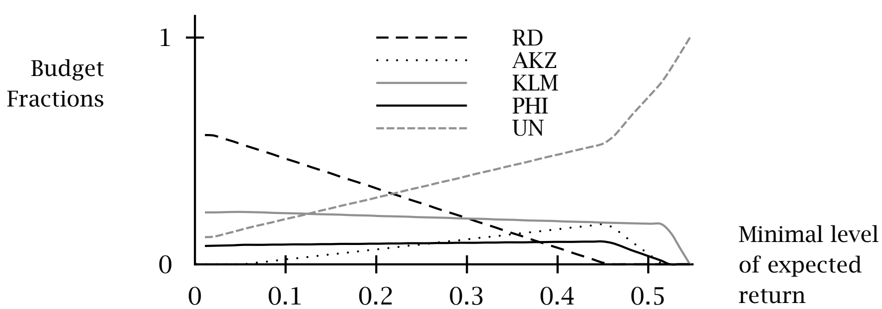


Figure 18.5: Portfolio diversification

## 18.8 One-sided variance as portfolio risk

*This section*

In this section, the notion of one-sided variance will be introduced as an extension of regular variance. Based on this new notion a more realistic reformulation of the basic Markowitz model can be developed.

*Main fallacy of variance as portfolio risk*

A serious drawback of using variance as a measure of risk is that it penalizes both high and low portfolio returns. In this sense, it fails to capture an investor's basic preference for higher rather than lower portfolio returns.

*Concept of one-sided variance*

The concept of one-sided variance is similar to the concept of variance but considers only deviations either below (downside variance) or above (upside variance) of some specific target value. For each random variable $R_j$ consider the following two definitions of one-sided variance with respect to the desired expected portfolio return $M$.

$$\text{DownVar}[R_j, M] = \text{E}[(\max[M - R_j, 0])^2] = \sum_{t \mid r_{tj} \le M} (M - r_{tj})^2 p(r_t)$$

$$\text{UpVar}[R_j, M] = \text{E}[(\max[R_j - M, 0])^2] = \sum_{t \mid r_{tj} \ge M} (r_{tj} - M)^2 p(r_t)$$

*Downside variance of a portfolio*

Reflecting the investor's preference for higher rather than lower portfolio returns, the focus in this section will be on downside variance of a portfolio as the risk measure to be minimized.

$$\begin{aligned} \text{DownVar}[\sum_j R_j x_j, M] &= \text{E}[(\max[M - \sum_j R_j x_j, 0])^2] \\ &= \sum_{t \mid \sum_j r_{tj} x_j \le M} [M - \sum_j r_{tj} x_j]^2 p(r_t) \end{aligned}$$

*Reformulation required*

The above expression makes reference to the unknown budget fractions $x_j$ inside the condition controlling the summation. Such expressions cannot be handled by current solution packages, as these require the structure of the constraints to be known and fixed prior to solving the model. That is why another representation must be found such that the special condition controlling the summation is no longer present.

*Introduce new variables*

Whenever you are required to reference positive and/or negative values of an arbitrary expression, it is convenient to write the expression as the difference between two nonnegative variables. This reformulation trick was already introduced in Chapter 6.

*Equivalent formulation*

As indicated before, the focus is on downside variance, and only one new variable needs to be introduced. Let the new variable $q_t \geq 0$ measure the below deviations of the target value $M$. Then,

$$\textbf{Minimize} \sum_{t \mid \sum_j r_{tj} x_j \leq M} [M - \sum_j r_{tj} x_j]^2 p(r_t)$$

can be rewritten as

**Minimize:**

$$\sum_t p(r_t) q_t^2$$

**Subject to:**

$$M - \sum_j r_{tj} x_j \leq q_t \qquad \forall t$$

$$q_t \geq 0 \qquad \forall t$$

*Comment*

Note that this reformulation does not result in a simple computational formula, but in an optimization model with inequalities. The objective function will force the nonnegative $q_t$ variables to be as small as possible. This results in $q_t = M - \sum_j r_{tj} x_j$ whenever $M$ is greater than or equal to $\sum_j r_{tj} x_j$, and $q_t = 0$ otherwise.

*Summary of model using downside variance*

Based on the above development concerning the downside risk of a portfolio, the tactical quadratic optimization model of Section 18.6 can be rewritten with $x_j$ and $q_t$ as the decision variables.

**Minimize:**

$$\sum_t p(r_t) q_t^2$$

**Subject to:**

$$\sum_j r_{tj} x_j + q_t \geq M \qquad \forall t$$

$$\sum_j m_j x_j \geq M$$

$$\sum_j x_j = 1$$

$$x_j \geq 0 \qquad \forall j$$

$$q_t \geq 0 \qquad \forall t$$

*Global optimality guaranteed*

Global optimality of any optimal solution to the above model is guaranteed. As before, the quadratic and minimizing objective function possesses a positive semi-definite matrix of second-order derivatives, and all constraints are linear.

*Solution*

To compare the computational results for the two notions of variance, Table 18.3 presents the optimal budget fractions for a minimal level of expected return of $M = 0.2$. Note that a lower total risk value associated with down-sided variance does not necessarily imply a lower risk, because there is no ordinal relationship between both risk measures.

| | two-sided variance | down-sided variance |
|---|---|---|
| RD | 0.335 | 0.337 |
| AKZ | 0.066 | 0.046 |
| KLM | 0.214 | 0.351 |
| PHI | 0.091 | 0.017 |
| UN | 0.294 | 0.250 |
| total risk | 8.982 | 5.189 |

Table 18.3: Optimal budget fractions for $M = 0.2$

## 18.9 Adding logical constraints

*This section*

This section discusses several logical conditions that can be added to the portfolio selection models in this chapter. The resulting models are mixed-integer quadratic programming models. When linearized, these models can be successfully solved within AIMMS.

*Imposing minimum fractions*

Investing extremely small fractions of the budget in an investment category or individual security is unrealistic in real-life applications. A natural extension is to introduce an either-or condition. Such a condition specifies for each investment category or individual security to invest either at least a smallest positive fraction of the budget or nothing at all.

*Fixed fee for transaction costs*

Some financial institutions may charge a fixed fee each time they execute a transaction involving a particular type of security. Such a fee may have a limiting effect on the number of different securities in the optimal portfolio, and is significant enough in most real-world applications to be considered as a necessary part of the model.

*Conditional selections*

A portfolio manager may have specific preferences for various types of securities. Some of these preferences are of the form: if security of type A is to be included in the optimal portfolio, then also security of type B has to be included. Such conditional selections result from practical considerations, and therefore form a natural extension of the model.

*Use logical variables*

The logical conditions described in the previous paragraphs can be translated into new constraints and new variables to be added to the portfolio models developed thus far. None of the above logical conditions are worked out in detail in this chapter, as you have already encountered them in previous chapters. The formulation tricks involving binary decision variables are described in detail in Chapter 7 with additional illustrations thereof in Chapter 9.

*Motivation to linearize the objective*

Adding binary variables to the quadratic programming model of the previous section requires the availability of a solver for quadratic mixed-integer programming. One way to circumvent the need for this class of algorithms is to approximate the quadratic terms in the objective by piecewise linear functions, thus obtaining a linear formulation. Adding binary variables to that formulation causes the entire model to become a mixed-integer linear program, for which solvers are readily available.

## 18.10 Piecewise linear approximation

*This section*

In this section the piecewise approximation of the quadratic function $f(q_t) = q_t^2$ is explained in detail. Special attention is paid to the determination of the overall interval of approximation, the quality of the approximation, and the corresponding division into subintervals.

*Illustration of piecewise approximation*

Figure 18.6 illustrates how a simple quadratic function can be approximated through a piecewise linear function. The function domain is divided into equal-length subintervals. By construction, both the true function value and the approximated function value coincide at the endpoints of each subinterval. The slopes of the linear segments increase from left to right, which is what you would expect for a piecewise convex function. Through visual inspection you might already conclude that the approximation is worst at the midpoints of each subinterval. As will be shown, the size of the corresponding maximum approximation error is the same for each interval, as long as intervals are of equal length.

*Components piecewise formulation*

Recall from the previous section that the quadratic objective function to be minimized is $\sum_t p(r_t)q_t^2$. The individual quadratic terms $f(q_t) = q_t^2$ can each be approximated independently over a finite portion of $q_t$-axis divided into subintervals indexed with $t$ and $l$. The length of each subinterval is denoted

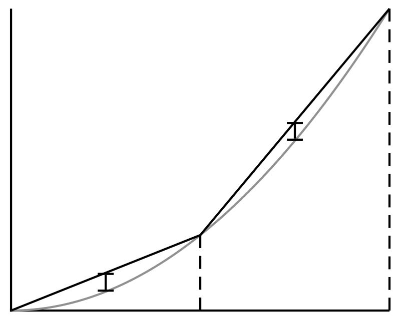

Figure 18.6: Piecewise linear approximation illustrated

with $\bar{u}_{tl}$. For each subinterval, a variable $u_{tl}$ is introduced where

$$0 \leq u_{tl} \leq \bar{u}_{tl} \qquad \forall (t, l)$$

In addition, the *slope* of the function $f(q_t)$ in each subinterval is defined as

$$s_{tl} = \frac{f(q^e_{tl}) - f(q^b_{tl})}{q^e_{tl} - q^b_{tl}} \qquad \forall (t, l)$$

where $q^b_{tl}$ and $q^e_{tl}$ denote the beginning and end values of the intervals, respectively. The following three expressions, defining the approximation of each individual term $f(q_t)$, can now be written.

$$\begin{aligned} f(q_t) &= \sum_l s_{tl} u_{tl} && \forall t \\ q_t &= \sum_l u_{tl} && \forall t \\ u_{tl} &\leq \bar{u}_{tl} && \forall (t, l) \end{aligned}$$

*Correctness*

The above approximation only makes sense if the variable $u_{tl} = \bar{u}_{tl}$ whenever $u_{t,l+1} > 0$. That is, $u_{tl}$ must be filled to their maximum in the left-to-right order of the intervals $l$, and no gaps are allowed. Fortunately, this condition is automatically guaranteed for convex functions to be minimized. The slope $s_{tl}$ increases in value for increasing values of $l$, and any optimal solution in terms of the $u_{tl}$-variables will favor the variables with the lowest $s_{tl}$-values.

*Function domain*

Recall that $q_t$ denotes downside variance, which is always greater than or equal to zero. The largest value that $q_t$ can attain is when $r_{tj}$ attains its smallest value over all investment categories or individual securities $j$, and the corresponding fraction $x_j$ is equal to one. It is highly unlikely that $x_j$ will be one, but this value establishes the proper interval size for $q_t$.

$$0 \leq q_t \leq \bar{q}_t \equiv \min_j r_{tj} \qquad \forall t$$

*The value of slope $s_{tl}$*

For the special case of quadratic terms $f(q_t) = q_t^2$, the general expression for the slope of the linear approximation

$$s_{tl} = \frac{f(q_{tl}^e) - f(q_{tl}^b)}{q_{tl}^e - q_{tl}^b} \qquad \forall (t,l)$$

reduces to the following simple expression in terms of endpoints.

$$s_{tl} = \frac{(q_{tl}^e)^2 - (q_{tl}^b)^2}{q_{tl}^e - q_{tl}^b} = \frac{(q_{tl}^e + q_{tl}^b)(q_{tl}^e - q_{tl}^b)}{q_{tl}^e - q_{tl}^b} = q_{tl}^e + q_{tl}^b \qquad \forall (t,l)$$

*Approximation is worst at midpoint . . .*

The function $q_t^2$ is convex, and the linear approximation on the interior of any subinterval of its domain overestimates the true function value. The point at which the approximation is the worst, turns out to be the midpoint of the subinterval. The steps required to prove this result are as follows. First write an error function that captures the difference between the approximated value and the actual value of $q_t^2$ on a particular subinterval. Then, find the point at which this error function attains its maximum by setting the first derivate equal to zero.

*. . . and can be derived as follows*

Consider the error function to be maximized with respect to $u_{tl}$

$$((q_{tl}^b)^2 + s_{tl}u_{tl}) - (q_{tl}^b + u_{tl})^2$$

By taking the first derivative with respect to $u_{tl}$ and equating this to zero, the following expression results.

$$s_{tl} - 2(q_{tl}^b + u_{tl}) = 0$$

Using the fact that $s_{tl} = q_{tl}^e + q_{tl}^b$, the value of $u_{tl}$ for which the above error function is maximized, becomes

$$u_{tl} = \frac{q_{tl}^e - q_{tl}^b}{2}$$

Note that a *maximum* occurs at this value of $u_{tl}$, because the second derivative of the error function is negative (a necessary and sufficient condition). As a result, the maximum is attained at the midpoint of the subinterval.

$$q_{tl}^b + u_{tl} = q_{tl}^b + \frac{q_{tl}^e - q_{tl}^b}{2} = \frac{q_{tl}^e + q_{tl}^b}{2}$$

*Maximum approximation error*

The size of the maximum approximation error $\boldsymbol{\epsilon}_{tl}$ can be determined in a straightforward manner by substituting the optimal $u_{tl}$ expression in the error function. This requires some symbolic manipulations, but finally results in the following simple compact formula.

$$\boldsymbol{\epsilon}_{tl} = \frac{(q_{tl}^e - q_{tl}^b)^2}{4}$$

*Number of subintervals*

Note that the above maximum approximation error is a function of the *length* of the subinterval, and is in no way dependent on the *position* of the interval. This implies that the choice of equal-length subintervals is an optimal one when you are interested in minimizing the maximum approximation error of the piecewise linear approximation of a quadratic function. In addition, the number of subintervals $n_t$ dividing the overall interval $[0, \bar{q}_t]$ can be determined as soon as the desired value of an overall $\boldsymbol{\epsilon}$, say $\bar{\boldsymbol{\epsilon}}$, is specified by the user of the model. The following formula for the number of subintervals $n_t$ of equal size guarantees that the maximum approximation error of $q_t$ will never be more than $\bar{\boldsymbol{\epsilon}}$.

$$n_t = \left\lceil \frac{\bar{q}_t}{2\sqrt{\bar{\boldsymbol{\epsilon}}}} \right\rceil$$

*The piecewise linear program*

Using the notation developed in this section, the following piecewise linear programming formulation of the portfolio selection model from the previous section can be obtained.

**Minimize:**

$$\sum_t p(r_t) \sum_l s_{tl} u_{tl}$$

**Subject to:**

$$\begin{aligned} \sum_j r_{tj} x_j + \sum_l u_{tl} &\geq M \qquad && \forall t \in T \\ \sum_j m_j x_j &\geq M && \\ \sum_j x_j &= 1 && \\ x_j &\geq 0 && \forall j \\ 0 \leq u_{tl} &\leq L_{tl} && \forall (t, l) \end{aligned}$$

## 18.11 Summary

In this chapter, both a strategic and a tactical portfolio selection problem have been translated into a quadratic programming model. The relatively small strategic model uses a covariance matrix as input, whereas the relatively large tactical model uses historic rates of return as scenarios to estimate the risk and expected return of a portfolio. Both models can be used to determine the particular combination of investment categories or securities that is the least risky for a given lower bound on expected return. Apart from single optimal solutions, parametric curves depicting the trade-off between risk and return were also provided. Several properties of the investment model were investigated. It was shown that $(a)$ any optimal solution is also a global optimum, $(b)$ the risk-reward curve is nondecreasing and convex, and $(c)$ multiple optimal portfolio returns are perfectly correlated. An improvement to the model was introduced by minimizing only downside risk, thus making the model more realistic. Further extensions were suggested to take into account such real-world requirements as minimum investment fractions, transaction costs and conditional security selections. Finally, a piecewise linear approximation of the quadratic objective function was introduced in order to keep the model with logical constraints within the framework of mixed-integer linear programming.

## Exercises

18.1 Implement the strategic investment model presented in Section 18.2 using the example data provided in Table 18.1. Use Aimms to reproduce the risk-reward curve illustrated in Figure 18.2.

18.2 Implement the tactical investment model presented in Section 18.6 using the example data presented in Table 18.2. Modify the objective function to represent downside variance as the measure of portfolio risk, and compare the result with the solution presented in Table 18.3.

18.3 Implement the piecewise linear formulation of the tactical investment model as described at the end of Section 18.10. Add the logical requirement that either at least 5% of the total budget is invested in any particular security or 0%. In addition, add the requirement that whenever the percentage invested in security 'RD' is greater than 20%, then the percentage invested in security 'KLM' has to be less than 30%. If the number of intervals becomes too large for a particular $t$ and a particular $\boldsymbol{\epsilon}$, design a dynamic scheme to adjust the interval length based on a previous solution.

# Bibliography


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[Re89] F.K. Reilly, *Investment analysis and portfolio management*, The Dryden Press, Orlando, Florida, 1989.