AIMMS Modeling Guide - Performance Assessment Problem

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AIMMS B.V.
Diakenhuisweg 29-35
2033 AP Haarlem
The Netherlands
Tel.: +31 23 5511512

AIMMS Inc.
11711 SE 8th Street
Suite 303
Bellevue, WA 98005
USA
Tel.: +1 425 458 4024

AIMMS Pte. Ltd.
55 Market Street #10-00
Singapore 048941
Tel.: +65 6521 2827

AIMMS
SOHO Fuxing Plaza No.388
Building D-71, Level 3
Madang Road, Huangpu District
Shanghai 200025
China
Tel.: ++86 21 5309 8733

Email: info@aimms.com
WWW: www.aimms.com

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Part IV

Intermediate Optimization Modeling Applications

Chapter 13

A Performance Assessment Problem

This chapter

In this chapter, you will encounter the problem of determining the performance of a set of comparable organizations. Such evaluation problems are nontrivial. The basic concept of relative efficiency of one organizational unit in relation to the other units is introduced. Based on this concept, the underlying problem can be translated into a collection of linear programming models using relative efficiency as an optimization criterion. Efficient organizations can then be identified, and form a reference for the other organizations. An example with seven input-output categories and 30 related organizations is provided for illustrative purposes.

References

The term Data Envelopment Analysis (DEA) is the general heading under which many papers on the assessment of comparable organizations have been written. The term was introduced [Ch78]. Since that time, several books on the topic have been written. One such references is [No91]. Unfortunately, neither the term Data Envelopment Analysis nor its abbreviation DEA creates an immediate mental picture when considering performance assessment of comparable organizations. For that reason, the term is not used any further in this chapter.

Keywords

Linear Program, Mathematical Reformulation, What-If Analysis, Worked Example.

13.1 Introduction and terminology

Decision making units (DMU's)

In large distributed organizations, senior management routinely wants to evaluate the relative performance of similar decision making units (DMU's) under their control. One example of such a distributed organization in the private sector is a bank with branch offices operating as autonomous DMU's. Another example is a retail chain with similar outlets as DMU's. In the public sector you may think of a Board of Education overseeing many schools, a health program governing several hospitals, or a state prison system managing their prisons.

Management issues

Senior management has specific objectives in mind when evaluating the organization's DMU's. Typical issues of concern in the private sector are increasing sales, reducing costs, identifying good performers, etc. Typical issues in the public sector are improving service levels, staff utilization and the management of large expenditures. Through performance evaluations, senior management gains insight into the operation of the individual DMU's under its control. For the case where the overall organization has to shrink in terms of the number of DMU's, these evaluations can be used as a basis for eliminating the truly poor performers.

Outputs and inputs

When measuring the performance of its DMU's, management should not limit the analysis to a few isolated measures such as profit or cost. Instead, a wide range of input and output factors should be considered in order to get a comprehensive insight into how well an organization is really performing in comparison to others.

Performance measures ...

For every type of application there are both specific and generic performance measures. Some of them are easy to measure, while others may be difficult to capture in quantitative terms.

... in the private sector

Performance measures encountered in private sector applications are often financial in nature. Typical examples are total revenue, revenue growth, uncontrollable cost, controllable costs, total liabilities, net capital employed, etc. Examples of non-financial performance measures are competition, age of unit, catchment population, customer service, pitch, etc.

... and in the public sector

Examples of performance measures in the public sector are staff utilization, productivity, throughput, accuracy, customer satisfaction, number of publications, client/staff ratio's, etc.

Absolute and relative efficiency

Efficiency can be described as the use made of resources (inputs) in the attainment of outputs. A DMU is 100% absolute efficient if none of its outputs can be increased without either increasing other input(s) or decreasing other output(s). A 100% relative efficiency is attained by a particular DMU once any comparison with other relevant DMU's does not provide evidence of inefficiency in the use of any input or output. In the sequel this concept of relative efficiency will be translated into a workable mathematical formulation.

13.2 Relative efficiency optimization

A ratio measure ...

In this chapter, the above relative efficiency measure of a DMU is defined mathematically by the ratio of a weighted sum of outputs to a weighted sum of inputs. This ratio can be maximized by allowing the best possible selection of nonnegative weights for each DMU separately. This implies the existence of as many optimization models as there are DMU's. Of course, the weights cannot be arbitrarily large, and thus have to be restricted as explained in this section.

... expresses relative efficiency

A DMU is said to be efficient relative to other DMU's if the value of its ratio efficiency measure is at least as large as those of the other DMU's using the same weights.

Verbal model

The following verbal model description expresses an optimization model for each DMU separately.

Maximize: *relative efficiency measure for a particular DMU,*
Subject to:
- *for all DMU's the corresponding relative efficiency measure is restricted to be less than or equal to 1.*

In the above formulation, the restriction of each ratio measure to be less than or equal to 1 is meant to indicate that both the numerator and the denominator are equally important in determining the relative efficiency.

Notation

The following symbols will be used.

Indices:

d	*decision making units*
i	*observed input categories*
j	*observed output categories*

Parameters:

a_{id}	*observed input level (>0) of input i for DMU d*
b_{jd}	*observed output level (>0) of output j for DMU d*
p	*element parameter with specific DMU as its value*

Variables:

x_{id}	*weight to be given to input i for DMU d*
y_{jd}	*weight to be given to output j for DMU d*

The objective for each DMU separately

The relative efficiency ratio measure is defined for each DMU separately. For this reason, the element parameter p (a standard concept in AIMMS) is used in the second index position of each identifier. As stated previously, the objective is to maximize the ratio of a weighted sum of outputs to a weighted sum of inputs. This can be written as follows.

$$\left(\sum_{j\in J} b_{jp}y_{jp}\right) \Big/ \left(\sum_{i\in I} a_{ip}x_{ip}\right)$$

Ratio constraint

The optimal selection of the nonnegative weights x_{ip} and y_{jp} are used to compare the performance of the other DMU's based on their values of the various input and output factors. By restricting the corresponding ratio to 1 for all DMU's (including DMU p), relative efficiency can be at most 1 (which can be interpreted as 100%).

$$\frac{\sum_{j \in J} b_{jd} y_{jp}}{\sum_{i \in I} a_{id} x_{ip}} \leq 1$$

Summary of first formulation

A first mathematical formulation of the model can be stated as follows.

Maximize:

$$\left(\sum_{j \in J} b_{jp} y_{jp}\right) \bigg/ \left(\sum_{i \in I} a_{ip} x_{ip}\right)$$

Subject to:

$$\frac{\sum_{j \in J} b_{jd} y_{jp}}{\sum_{i \in I} a_{id} x_{ip}} \leq 1 \qquad \forall d \in D$$

$$x_{ip} \geq 0 \qquad \forall i \in I$$

$$y_{jp} \geq 0 \qquad \forall j \in J$$

Some simple manipulations

In the previous formulation you may alter any optimal solution by multiplying the weight variables with a constant. Such multiplication does not alter the input-output ratio's. By forcing the weighted sum of inputs (i.e. the denominator) in the objective function to be equal to 1 you essentially remove this degree of freedom. In addition, the nonlinear ratio constraints can be transformed into linear constraints by multiplying both sides of the inequalities with their positive denominator. The denominator is always positive, because (a) all input and output levels are assumed to be positive, and (b) the nonnegative input weights cannot all be 0 when the weighted sum of inputs in the objective function is equal to 1.

Resulting linear program

The resulting linear programming formulation is now as follows.

Maximize:

$$\sum_{j \in J} b_{jp} y_{jp}$$

Subject to:

$$\sum_{j \in J} b_{jd} y_{jp} \leq \sum_{i \in I} a_{id} x_{ip} \qquad \forall d \in D$$

$$\sum_{i \in I} a_{ip} x_{ip} = 1$$

$$x_{ip} \geq 0 \qquad \forall i \in I$$

$$y_{jp} \geq 0 \qquad \forall j \in J$$

Concept of reference set

The optimal value of the objective function for the particular DMU referenced through p is either 1 or less than 1. In the latter case, there must be one or more other DMU's which have a relative efficiency equal to 1 based on these same weights. If this were not the case, all output weights could be multiplied with a scalar greater than 1. This increases the optimal value of the objective function, which is a contradiction in terms. The subset of other DMU's with relative efficiency equal to 1 is referred to as the *reference set* of p.

13.3 A worked example

Background

Consider a chain of 30 stores with total revenue of roughly US$ 72 million and total cost of roughly US$ 68 million. The overall profit-before-tax slightly exceeds 5.5%, and senior management considers this too low for their type of business. As a result, they decide to initiate a study to assess the performance of their stores. In particular, they would like to make an initial selection of those stores that are relatively poor performers. These stores can then be investigated further prior to making any decisions regarding selling, closing or improving one or more of these poor performers.

Available data

In Table 13.1 you will find the input and output factors for a total of 30 DMU's numbered (for simplicity) from 1 to 30. The number of factors is kept small in this example, but in real applications you may encounter several additional factors not mentioned here. The two main factors determining profit are 'Revenue' and 'Total Cost', measured in 1000's of dollars. The total cost figures have been split into 'Staff Cost' (variable cost) and 'Non-staff Cost' (fixed cost). The three non-financial performance measures are 'Customer Service', 'Competition' and 'Age of Store'. Customer service is expressed as a rating between 1 (lowest) and 10 (highest). Competition is a count of the number of competitors within a fixed driving distance. The age of a store is expressed in terms of months.

Management versus DMU's

The study is initiated by senior management, and they control the way that the assessment is performed by choosing the input and output factors to be considered. As will be illustrated, such a choice will have a definite impact on the results and their conclusions. On the other hand, the weights associated with each of the factors are chosen to optimize the relative efficiency of each

	Input-Output Factors						
	Total Cost [10^3 \$/yr]	Staff Cost [10^3 \$/yr]	Non-staff Cost [10^3 \$/yr]	Age of Store [month]	Competition [-]	Customer Service [-]	Revenue [10^3 \$/yr]
DMU-01	1310	238	1072	18	11	8	1419
DMU-02	2091	459	1632	46	12	8	3064
DMU-03	930	154	776	36	9	5	987
DMU-04	3591	795	2796	34	9	7	3603
DMU-05	2729	571	2158	35	13	9	2742
DMU-06	2030	497	1533	57	7	6	2536
DMU-07	3247	558	2689	36	5	9	4320
DMU-08	2501	571	1930	40	12	7	3495
DMU-09	2299	407	1892	17	8	8	2461
DMU-10	2413	306	2107	23	16	5	1851
DMU-11	1450	458	992	49	18	10	1935
DMU-12	2758	494	2264	35	9	8	3558
DMU-13	1857	360	1497	59	25	5	2088
DMU-14	3195	618	2577	51	9	8	3963
DMU-15	3505	759	2746	38	12	9	3918
DMU-16	1408	313	1095	24	9	8	1693
DMU-17	1127	253	874	17	5	7	1196
DMU-18	1637	340	1297	21	13	7	1945
DMU-19	2305	551	1754	27	7	6	3207
DMU-20	1781	303	1478	34	29	4	1622
DMU-21	3122	642	2480	26	5	10	2334
DMU-22	2597	465	2132	20	11	6	1387
DMU-23	1817	335	1482	28	4	9	1969
DMU-24	3483	825	2658	53	11	6	3422
DMU-25	1954	424	1530	11	15	3	1189
DMU-26	1120	159	961	4	5	6	810
DMU-27	1408	248	1160	36	7	3	1081
DMU-28	3420	672	2748	44	7	9	3088
DMU-29	2242	364	1878	18	11	5	1796
DMU-30	2643	490	2153	27	6	7	3243

Table 13.1: Observed values per factor

particular DMU separately. It is thus possible that a particular weight can be zero, thereby eliminating the effect of the corresponding factor relative to the other factors. If a DMU with its own optimal weights cannot be 100% relative efficient, then it is not part of the reference set and thus subject to further scrutiny.

Output versus input

In each application a decision must be made as to which factors are output and which factors are input. In general, an output factor is a factor that refers to aspects of achievement, while an input factor is a factor that aids or hinders the production of the outputs. In this example, 'Revenue' and 'Customer

Service' are achievement factors, while all other categories are considered as input factors. A priori, the category 'Age of Store' cannot be seen as always hindering or always aiding any of the achievement factors. As was indicated in the previous paragraph, it is senior management who decides on the particular assessment experiments, and they will only consider those inputs and outputs that are meaningful to them and their decisions.

Correlated data

In practical applications the number of input and output factors can be quite large. In order to make meaningful choices for assessment, it is customary to examine the correlations between the factors. The reasons to exclude a certain factor from consideration may be the fact that another factor is already considered between which there exists a high correlation. Consider the correlations in Table 13.2. As it turns out, there is a high correlation between 'Total Cost', 'Staff Cost' and 'Non-staff Cost'. In addition, there is a dependency between these cost categories, because the latter two add up to the first. This dependency shows up in the third experiment where all factors are considered and the optimal weight associated with 'Total Cost' is zero for each DMU.

	Total Cost	Staff Cost	Non-staff Cost	Age of Store	Competition	Customer Service	Revenue
Total Cost	1.000	0.916	0.994	0.331	−0.134	0.355	0.826
Staff Cost		1.000	0.865	0.466	−0.110	0.410	0.822
Non-staff Cost			1.000	0.283	−0.137	0.329	0.802
Age of Store				1.000	0.265	0.156	0.511
Competition					1.000	−0.343	−0.162
Customer Service						1.000	0.484
Revenue							1.000

Table 13.2: Correlations between the factors

First experiment

In this example only three experiments are discussed. In practice, a large number of experiments will be performed. The first experiment uses the factor 'Revenue' as output and the factor 'Total Cost' as input. All other factors are ignored. In this experiment, the objective function is therefore a simple ratio, and you would expect only the most profitable DMU to be 100% relative efficient. This is indeed the case, and the reference set consists of DMU-02. The seven poor performers (starting from the worst) are DMU-22, DMU-25, DMU-26, DMU-21, DMU-10, DMU-27, and DMU-29.

Second experiment

The second experiment does not focus exclusively on 'Revenue' as output, but also considers 'Customer Service' as output. After all, high customer service may lead to improved sales in the future. 'Total Cost' remains the only input. In this experiment, DMU-11 joins DMU-02 as 100% relative efficient. Note that when factors are added to an experiment and no factors are deleted, then no DMU's leave the reference set and only new ones can enter. One of the seven poor performers, namely DMU-26, has improved its position relative to other DMU's. The order of the poor performers has also changed. The seven poor performers (starting from the worst) are now DMU-22, DMU-25, DMU-10, DMU-27, DMU-21, DMU-29, and DMU-20.

Third experiment

In the third and last experiment, both 'Revenue' and 'Customer Service' are considered as output, while all other factors are used as input. Such an experiment offers each DMU plenty of opportunity to improve its relative efficiency. As a result, twelve DMU's have become 100% relative efficient and form the reference set illustrated in Table 13.3. There is also some movement in the set of poor performers. Starting from the worst, they are DMU-22, DMU-27, DMU-25, DMU-24, DMU-20, DMU-28, and DMU-29.

	Relative efficiency		Relative efficiency		Relative efficiency
DMU-01	1.0000	DMU-23	1.0000	DMU-10	0.8204
DMU-02	1.0000	DMU-26	1.0000	DMU-13	0.8106
DMU-03	1.0000	DMU-08	0.9841	DMU-05	0.7999
DMU-07	1.0000	DMU-30	0.9657	DMU-29	0.7360
DMU-09	1.0000	DMU-18	0.9652	DMU-28	0.7342
DMU-11	1.0000	DMU-12	0.9612	DMU-20	0.7099
DMU-16	1.0000	DMU-06	0.9339	DMU-24	0.7046
DMU-17	1.0000	DMU-14	0.9032	DMU-25	0.6876
DMU-19	1.0000	DMU-15	0.8560	DMU-27	0.5990
DMU-21	1.0000	DMU-04	0.8372	DMU-22	0.5271

Table 13.3: Optimal relative efficiencies with all factors considered

Conclusions

On the basis of these few experiments, senior management can only make some preliminary conclusions. It is certainly true that DMU-22 has been consistently the worst performer of all. In addition, the three poorest performers have been so in all three experiments, and DMU-29 has always been on the edge of being a poor performer. However, the extent to which any of these results can be interpreted in a context which is relevant to managing the organization, is not clear at this point. In practice, assessment questions can be analyzed through the type of mathematical and statistical analysis described in this chapter, but extensive and detailed subsequent analysis of selected DMU's is required before any sound decision by senior management regarding closure, investment, target setting, etc. can be made.

13.4 Computational issues

This section

In this section you will find some observations regarding the quality of the numerical solution for the type of assessment model discussed in this chapter.

Numerical difficulties

Computational characteristics of the mathematical performance assessment model described in this chapter have been studied in the literature ([Ch96]). Numerical difficulties have been observed when

- there are many DMU's,
- the number of inputs and outputs is large,
- the inputs and outputs are of different orders of magnitude, and
- the data sets for some DMU's are nearly identical.

It is not within the scope of this book to explain why these difficulties occur. Fortunately, there are some simple precautions you can take to reduce the likelihood of any numerical difficulties. Why these precautions may have a positive effect on the numerical quality of your solution is again outside the scope of this book.

Precautionary measures

If you want to be on the safe side, follow the two precautionary measures recommended in this paragraph. Their implementation will never lead to a deterioration of solution quality. The first precautionary measure is to scale your data such that the values of each factor center around the same constant value, say 1. The effect of this measure is that the weights will tend to assume the same *relative* order of magnitude. The second measure is to change the right-hand side of the constraint limiting the weighted input. Instead of using the value 1, you may want to use the total number of input factors used in the experiment. This will cause the *absolute* size of the weights to be of order 1. In this case the relative efficiency of each DMU is no longer measured as a fraction but as a multiple thereof. It is straightforward to implement these precautionary measures in AIMMS using the `Unit` attribute associated with parameters.

13.5 Summary

In this chapter a general framework for assessing the relative performance of multiple decision making units has been presented. The analysis uses the concept of relative efficiency. The approach is balanced in that senior management is allowed to construct the various assessment experiments, while each particular decision making unit gets its own optimal weights for each of the input-output factors selected for the experiments. The corresponding model is a linear program to be solved for each decision making unit. Some suggestions to improve numerical performance were made. A worked example with seven input-output categories and thirty related decision making units was provided for illustrative purposes.

Exercises

13.1 Implement the mathematical program described at the end of Section 13.2 using the example data provided in Section 13.3. Repeat the three experiments described in Section 13.3, and observe the efficiency associated with the DMU's.

13.2 Implement the precautionary measures described in Section 13.4 as part of your model in Aimms. Write a procedure that will work for any data set related to the input-output categories used in Section 13.3.

13.3 Can you explain for yourself why the optimal weight associated with the category 'Total Cost' in the third experiment is zero for each DMU?

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